A note on the distribution of the maximum of a set of Poisson random variables

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Given a set of independent Poisson random variableswith common mean, we study the distribution of their maximum and obtain an accurate asymptotic formula to locate the most probable value of the maximum. We verify our analytic results with very precise numerical computations.

We deal with a set of independent Poisson random variables $\{X_1, X_2, \ldots, X_n\}$ with common mean λ, so that $\Pr[X_i = k] = e^{-\lambda}\lambda^k/k!$. We let $M_n = \max(X_i)$ and wish to describe the distribution of M_n. Our motivation is a problem in random graph theory, where we were interested in the distribution of maximum degree in graphs with Poisson degree distribution.

We have

$$\Pr[X_i < k] = Q(k, \lambda) \equiv \Gamma(k, \lambda)/\Gamma(k)$$

where Q and $\Gamma(\cdot,\cdot)$ are incomplete Gamma functions; that is,

$$\Gamma(a, x) = \int_x^\infty t^{a-1}e^{-t}\mathrm{d}t.$$

From the independence of the Poisson variables,

$$\Pr[M_n \leqslant k] = \Pr[X_1 \leqslant k]^n = Q(k+1, \lambda)^n = \Gamma(k+1, \lambda)^n/\Gamma(k+1)^n.$$

Our aim is to approximate the distribution of M_n. We have

$$\begin{aligned}\Pr[M_n = k] &= \Pr[M_n \leqslant k] - \Pr[M_n \leqslant k-1] \\ &= Q(k+1, \lambda)^n - Q(k, \lambda)^n\end{aligned}$$

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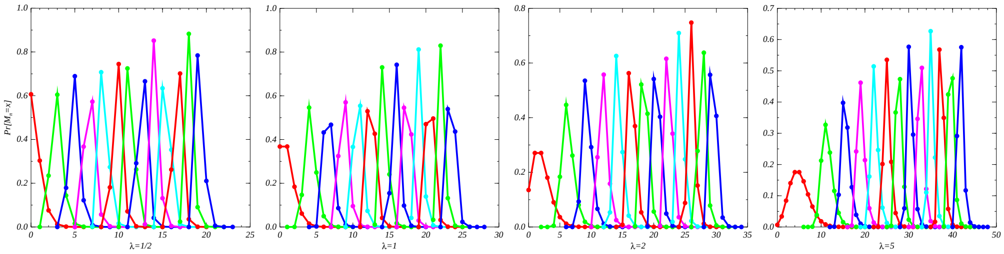

Figure 1: From left to right: the distribution of the maximum of Poisson variables for $\lambda = 1/2, 1, 2, 5$ (left to right) and $n = 10^0, 10^2, 10^4, \ldots, 10^{24}$. Note that there is an error in Fig 1 in Anderson et al. (1997), where the curves labelled $k = 6$ and $k = 8$ are incorrect.

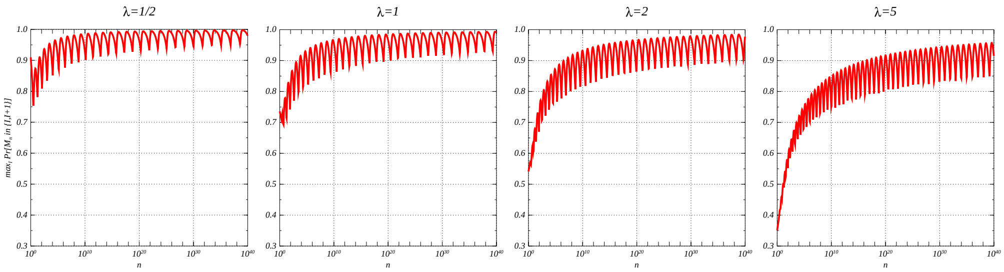

Figure 2: The maximal probability (with respect to I_n) that $M_n \in \{I_n, I_n{+}1\}$ for $\lambda = 1/2, 1, 2, 5$ (left to right) and $10^0 \leqslant n \leqslant 10^{40}$. The curves show the probability that M_n takes either of its two most frequently occurring values.

Examples of these distributions are shown in Figure 1. These numerical results demonstrate the so-called *focussing* effect; the maxima M_n are concentrated on at most two adjacent integers for large n; we call them *modal values*. It is this focussing that allows us to characterize the distributions very precisely by a single asymptotic estimate.

In previous work on this problem, Anderson (1970) proved the existence of integers I_n such that $\Pr[M_n \in (I_n, I_n{+}1)] \to 1$ as $n \to \infty$ for fixed $\lambda > 0$; and that $I_n \sim \beta_n$, where β_n is defined as the unique solution of $Q(\beta_n, \lambda) = 1/n$.

Following this work, Kimber (1983) computed an asymptotic result; he showed $I_n \sim \log n/\log\log n$ and $P_n \sim (k/I_n)^{1+B_n}$ with B_n dense in $[-1/2, 1/2]$, and concluded that to the first order, the rate of growth of I_n is independent of the Poisson parameter λ. He concluded that P_n, defined as $P_n = \Pr[M_n \in (I_n, I_n{+}1)]$, oscillates and the oscillation persists for arbitrar-

ily large n. We illustrate in Figure 2 exactly how this probability oscillates as $n \to \infty$. Our numerical experiments show that $\log n / \log\log n$ estimates I_n very poorly. We aim to improve this asymptotic formula.

Our method is a refinement of that of Kimber; that is, we consider a continuous distribution g which interpolates the Poisson maximum distribution, and we solve $g(x) = 1/n$. Consider $g_\lambda(x) \equiv 1 - \Gamma(x+1, \lambda)/\Gamma(x+1)$ for fixed $\lambda \in \mathbb{R}^+$, which is a strictly decreasing function on $(0, \infty)$. If $\epsilon = 1/n$ is a small positive real, then $g_\lambda(x)$ has a unique root $x(\epsilon)$ which increases as $\epsilon \to 0^+$. We will develop an asymptotic expansion (as $\epsilon \to 0$) of this root $x(\epsilon)$.

We have

$$g_\lambda(x) = \exp(-\lambda)\ \lambda^x \sum_{i=1}^{\infty} \frac{\lambda^i}{\Gamma(x+i+1)}$$

and we will work with

$$\log(g_\lambda(x)) = -x\log(x) + (1+\log\lambda)x - \frac{3}{2}\log(x) + \left(\log\lambda - \lambda - \frac{\log(2\pi)}{2}\right) + \frac{\lambda - 13/12}{x} + \mathcal{O}(x^{-2}). \quad (1)$$

A first approximation to the solution of $\log(g_\lambda(x)) = -\log n$ large and negative is given by keeping only the dominant first two terms in Equation (1):

$$M_n \sim x_0 \equiv \frac{\log n}{W\left(\frac{\log n}{\exp(1)\lambda}\right)},$$

where $W(\cdot)$ is the principal branch of Lambert's W function (Corless et al., 1996). That this is already quite accurate can be seen form the dark blue curves in Figure 3. However, we would like to do better; ideally the error should be less than unity so that the mode of the distribution is correctly identified. A refinement x_1 may be generated by making a single Newton correction step; that is, $x_1 = x_0 - (h(x_0) + \log n)/h'(x_0)$, where h is some approximation to $\log(g_\lambda)$. For example, by keeping all terms in $\log(g_\lambda(x))$ and $\log(g_\lambda(x))'$ which do not vanish as $n \to \infty$, we obtain

$$M_n \sim x_1 = x_0 + \frac{\log\lambda - \lambda - \log(2\pi)/2 - 3\log(x_0)/2}{\log(x_0) - \log\lambda}.$$

This appears to have error less than unity for all values of n and λ considered in Figure 3, and so is probably sufficiently precise for all practical purposes. If

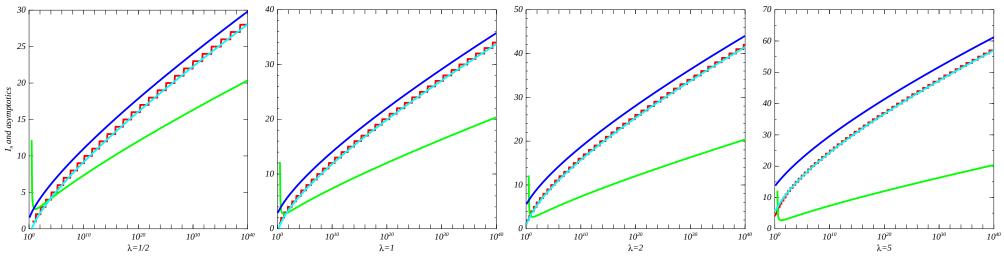

Figure 3: Exact values and asymptotics of I_n for $\lambda = 1/2, 1, 2, 5$ (left to right) and $n = 10^0, \ldots, 10^{40}$. The staircase red line (almost hidden by the cyan line) represents the exact mode I_n; the other lines represent asymptotic approximations: green for the result of Kimber (1983) (which is independent of λ), dark blue and cyan for our new results x_0 and x_1 respectively. The cyan curve always sits between the steps of I_n, meaning that x_1 has error less than unity.

further accuracy is needed, it may be obtained by additional Newton steps. In any case, both x_0 and x_1 are considerably more precise than Kimber's approximation.

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